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Application of the Darboux transformation to inverse problems with variable spectral parameters

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Abstract. We show that Darboux transformations can be used to find explicit solutions of deformed equations which have recently been found through the inverse method with a variable spectral parameter. We construct in particular the soliton solution of the deformed Maxwell–Bloch equations, and analyse its asymptotic behaviour.

1. Introduction

The inverse method of solving nonlinear partial differential equations was generalized in [1] to the case of a variable spectral parameter. This extension of the usual inverse method gave rise [1] to a new series of completely integrable nonlinear equations, and to interesting integrable deformations of known nonlinear equations‡. Reference [1] did not provide, however, any practical method of integrating the new equations. In this article [4] we therefore propose a simple algebraic method of integration which is applicable to the new deformed equations. The method we use here is a modified version [4] of the Darboux transformation (DT) method which has been successfully applied to a number of completely integrable equations [2–5, 9, 10]. An alternative method based on a finitely-many gaps integration has been proposed in [6].

In the inverse method the nonlinear equation to be solved appears as the compatibility condition of an auxiliary linear problem

$$\psi_x = U\psi \qquad \psi_t = V\psi \tag{1.1}$$

where $\psi_x \equiv (\partial/\partial x)\psi$ etc, and matrices U and V have only simple poles in the complex λ-plane,

$$\begin{aligned} U(\lambda,x,t) &= U_0(x,t) + \sum_{n=1}^{N_1} \frac{U_n(x,t)}{\lambda-\nu_n} \\ V(\lambda,x,t) &= V_0(x,t) + \sum_{n=1}^{N_2} \frac{V_n(x,t)}{\lambda-\mu_n} \end{aligned} \tag{1.2}$$

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‡ It is worth mentioning that the generalization of the traditional inverse problem method for the non-isospectral scattering problems have been studied in [11] and [12]. In this article we consider the case of both space and time dependence of the spectral parameter.

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The spectral parameter λ can now depend on space and time variables x, t, and on a hidden spectral parameter $z : \lambda = \lambda(x,t,z)$. Also the poles of the matrices U and V are dependent on x and t: $\nu_n = \nu_n(x,t)$, $\mu_n = \mu_n(x,t)$. The spectral parameter λ is found [1] to satisfy

$$\lambda_t = -\sum_{n=1}^{N_2} \frac{b_n}{\lambda - \mu_n} \qquad \lambda_x = -\sum_{n=1}^{N_1} \frac{c_n}{\lambda - \nu_n} \tag{1.3}$$

and therefore the poles ν_n, μ_n satisfy [1]

$$\nu_{n_t} = -\sum_{m=1}^{N_2} \frac{b_m}{\nu_n - \mu_m} \qquad \mu_{n_x} = \sum_{m=1}^{N_1} \frac{c_n}{\mu_n - \nu_m} \tag{1.4a}$$

where

$$c_{n_t} = 2c_n \sum_{m=1}^{N_2} \frac{b_m}{(\nu_n - \mu_m)^2} b_{n_x} = 2b_n \sum_{m=1}^{N_1} \frac{c_m}{(\mu_n - \nu_m)^2}. \tag{1.4b}$$

The compatibility condition of system (1.1) is

$$U_t - V_x + [U, V] = 0. \tag{1.5}$$

We demand this condition to be satisfied identically for λ and the linearly independent factors $(\lambda - \nu_n)^{-1}$ and $(\lambda - \mu_n)^{-1}$. Derivatives with respect to the spectral parameter in (1.5) will result in new terms as compared with those appearing in the usual inverse method.

We shall next formulate the DT method of integration. Let $(\psi_j, \phi_j)^{\mathrm{T}}$ be the solution of system (1.1) corresponding to the spectral parameter $\lambda_j \equiv \lambda(x,t,z_j)$; $j = 1, 2$. If we define the matrices Λ, Φ, P_n and Q_n such that

$$\begin{gathered} \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \Phi = \begin{pmatrix} \psi_1 & \psi_2 \\ \phi_1 & \phi_2 \end{pmatrix} \\ P_n = \begin{pmatrix} (\lambda_1 - \nu_n)^{-1} & 0 \\ 0 & (\lambda_2 - \nu_n)^{-1} \end{pmatrix} \qquad Q_n = \begin{pmatrix} (\lambda_1 - \mu_n)^{-1} & 0 \\ 0 & (\lambda_2 - \mu_n)^{-1} \end{pmatrix} \end{gathered} \tag{1.6}$$

then the system (1.1) can be expressed in the form

$$\begin{aligned} \Phi_x &= U_0 \Phi + \sum_{n=1}^{N_1} U_n \Phi P_n \\ \Phi_t &= V_0 \Phi + \sum_{n=1}^{N_2} V_n \Phi Q_n \end{aligned} \tag{1.7}$$

The DT method is based on the fact that system (1.7) is covariant with respect to linear transformations $\Phi \to \tilde{\Phi} = \mathcal{C}[\Phi]$. Also the transformed matrices $\tilde{U}_n$, $\tilde{V}_m$; $n = 0, 1, \ldots, N_1$, $m = 0, 1, \ldots, N_2$, will satisfy equation (1.5).

We consider a particularly simple ansatz for the linear transformation $\mathcal{C}$, for which

$$\tilde{\Phi} = S_0\Phi\Lambda + S_1\Phi \tag{1.8}$$

where matrices S_0 and S_1 act as free parameters. Under transformation (1.8) equations (1.7) should transform by covariance into

$$\tilde{\Phi}_x = \tilde{U}_0\tilde{\Phi} + \sum_{n=1}^{N_1}\tilde{U}_n\tilde{\Phi}P_n \quad \tilde{\Phi}_t = \tilde{V}_0\tilde{\Phi} + \sum_{n=1}^{N_2}\tilde{V}_n\tilde{\Phi}Q_n \tag{1.9}$$

By substituting (1.8) into (1.9) and demanding that the resulting equations should be satisfied identically for $\Phi\Lambda^n$, ΦP_n and ΦQ_n, we arrive at equations

$$\begin{aligned} &\sum_{n=1}^{N_1} S_0U_n + S_{1x} + S_1U_0 = \tilde{U}_0S_1 + \sum_{n=1}^{N_1}\tilde{U}_nS_0 \\ &S_{0x} + S_0U_0 = \tilde{U}_0S_0 \\ &(\nu_nS_0 + S_1)U_n - c_nS_0 = \tilde{U}_n(\nu_nS_0 + S_1) \qquad n = 1,\cdots,N_1 \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &\sum_{n=1}^{N_2} S_0V_n + S_{1t} + S_1V_0 = \tilde{V}_0S_1 + \sum_{n=1}^{N_2}\tilde{V}_nS_0 \\ &S_{0t} + S_0V_0 = \tilde{V}_0S_0 \\ &(\mu_nS_0 + S_1)V_n - b_nS_0 = \tilde{V}_n(\mu_nS_0 + S_1) \qquad n = 1,\cdots,N_2 \end{aligned} \tag{1.11}$$

In deriving equations (1.10) and (1.11) we have used equation (1.3) and the simple relationships

$$P_n\Lambda = \mathrm{I} + \nu_nP_n \qquad Q_n\Lambda = \mathrm{I} + \mu_nQ_n. \tag{1.12}$$

The matrices S_0 and S_1 are fixed by specific choices for the linear transformation $\mathcal{C}$. We consider here the case $\tilde{\Phi} = 0$, which is satisfied for specific solutions of the auxiliary linear problem, $\Phi = \Phi_0$, $\Lambda = \Lambda_0$,

$$\Phi_0 = \begin{pmatrix} \psi_1^0 & \psi_2^0 \\ \phi_1^0 & \phi_2^0 \end{pmatrix} \qquad \Lambda_0 = \begin{pmatrix} \lambda(x,t,z_1^0) & 0 \\ 0 & \lambda(x,t,z_2^0) \end{pmatrix} \tag{1.13}$$

where linearly independent $(\psi_j{}^0, \phi_j{}^0)^{\mathrm{T}}$, $j = 1, 2$ are to be chosen. The condition $\tilde{\Phi} = 0$ means simply that, through (1.8),

$$S_1 = -S_0\Phi_0\Lambda_0\Phi_0^{-1} \tag{1.14}$$

and S_0 will be defined by normalization which is to be chosen for each particular system. For a given seed solution $\{U_n, V_n\}$ of (1.5), and the corresponding matrix function Φ_0, equations (1.10) and (1.11) together with (1.8) and (1.14) will provide a new solution $\{\tilde{U}_n, \tilde{V}_n\}$ of (1.5), and, at the same time, a solution of equation (1.9).

2. Deformed Maxwell–Bloch equations

The system of Maxwell–Bloch (MB) equations [7],

$$E_\eta = \rho \qquad \rho_\xi = \mathcal{N}E$$
$$\mathcal{N}_\xi + \tfrac{1}{2}(E\bar\rho + \overline{E}\rho) = 0 \tag{2.1}$$

has been used to for example describe self-induced transparency (SIT) of a dispersive medium for coherent pulse propagation. Here $\bar\rho, \overline{E}$ denote the complex conjugate of ρ, E. In SIT E is the complex electric field amplitude, ρ is the induced (complex) polarization and $\mathcal{N}$ is the normalized population inversion of the medium.

In [1] a number of completely integrable deformations of equation (2.1) were reported; we consider here one of physical interest, i.e.

$$E_\eta = \rho \qquad \rho_\xi = \mathcal{N}E - \tilde c\rho$$
$$\mathcal{N}_\xi + \tfrac{1}{2}(E\bar\rho + \overline{E}\rho) = 4c - \tilde c\mathcal{N}. \tag{2.2}$$

Here functions c and $\tilde c$ are determined by equations

$$\tilde c_\eta = 0 \qquad c_\xi + 2c\tilde c = 0. \tag{2.3}$$

We may think of the deformed equation (2.2) as describing a coherent pulse propagation in a dispersive medium consisting of two-level atoms, including relaxation rate $\tilde c$ and source of inversion $4c$ The relaxation rate and source of inversion are connected in a non-trivial way through equation (2.3).

The linear system associated with equation (2.2) is [1]

$$\Phi_\xi = U_1\Phi\Lambda + U_0\Phi \qquad \Phi_\eta = \hat\rho\Phi P \tag{2.4}$$

where

$$U_1 = \sigma_3 \qquad U_0 = \tfrac{1}{2}\begin{pmatrix} 0 & E \\ -\overline{E} & 0 \end{pmatrix} \qquad \hat\rho = \begin{pmatrix} n & -\rho \\ -\bar\rho & -\tilde n \end{pmatrix}$$
$$\Lambda = \begin{pmatrix} \lambda(\xi,\eta,z_1) & 0 \\ 0 & \lambda(\xi,\eta,z_2) \end{pmatrix} \qquad P = (4\Lambda)^{-1}$$
$$\mathcal{N} = \tfrac{1}{2}(n + \tilde n) \tag{2.5}$$

(σ_3 is Pauli's matrix).

The matrices Λ and P satisfy [1] equations

$$\Lambda_\eta = 4cP \qquad P_\xi = \tilde cP. \tag{2.6}$$

We can now apply to system (2 2)–(2 6) the DT method described in section 1. In this way we find

$$S_0U_1 = U_1S_0$$
$$\tilde U_0 = S_1U_0S_1^{-1} + S_{1\xi}S_1^{-1} \tag{2.7a}$$

$$\tilde U_0 = S_0U_0S_0^{-1} + [S_1, U_1]S_0^{-1} + S_{0\xi}S_0^{-1} - \tilde cI$$
$$S_{0\eta} = 0 \tag{2.7b}$$

$$\tilde{\hat\rho} = S_1\hat\rho S_1^{-1} + 4cS_0S_1^{-1} \tag{2.8a}$$

$$\tilde{\hat\rho} = S_0\hat\rho S_0^{-1} + 4S_{1\eta}S_0^{-1}. \tag{2.8b}$$

This set of equations corresponds to equations (1.10)–(1.11) If we consider the case when $\tilde{c}$ is constant, then equation (2 7*b*) gives immediately the solution for S_0,

$$S_0 = S_0(\xi) = \mathrm{I} \;\; \mathrm{e}^{\tilde{c}\xi}. \tag{2.9}$$

In order to proceed further we use the fact that, acting on the set of solutions of the linear system (2.4), there is a group of automorphisms which has the generating transformation

$$\begin{pmatrix} \psi(\lambda) \\ \phi(\lambda) \end{pmatrix} \longrightarrow \begin{pmatrix} -\bar{\phi}(-\bar{\lambda}) \\ \bar{\psi}(-\bar{\lambda}) \end{pmatrix}. \tag{2.10}$$

Transformation (2.10) allows us to take the seed solution (1.13) in the form

$$\Phi_0 = \begin{pmatrix} \psi_1(\lambda_1) & -\bar{\phi}_1(-\bar{\lambda}_1) \\ \phi_1(\lambda_1) & \bar{\psi}_1(-\bar{\lambda}_1) \end{pmatrix} \tag{2.11}$$

It can be easily seen by differentiating $\det \Phi_0$ with respect to ξ and η that this determinant can be chosen to be non-zero and, therefore, the matrix Φ_0 is invertible for $\lambda = \lambda_1$. The corresponding matrix spectral parameter Λ is now

$$\Lambda_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\bar{\lambda}_1 \end{pmatrix} \tag{2 12}$$

where $\lambda_1 \equiv \lambda(\xi, \eta, z_1)$, and z_1 is the value of the hidden spectral parameter which specifies the seed solution.

Equations (2.3) and (2.6) are easily solved for constant $\tilde{c}$ to give

$$\begin{aligned} c &= c_0 \mathrm{e}^{-2\tilde{c}\xi} \\ \lambda_1 &\equiv \lambda(\xi, n, z_1) = (2c_0\eta + z_1)^{1/2} \mathrm{e}^{-\tilde{c}\xi}. \end{aligned} \tag{2.13}$$

Equations (2.7*b*) and (2.8*b*) can now be solved in terms of (2.11) and (2 12) In this way we find the Darboux dressing expressions for the electric field amplitude E, polarization ρ and inversion $\mathcal{N}$,

$$\tilde{E} = E + 4(\lambda_1 + \bar{\lambda}_1) \frac{\psi_1 \bar{\phi}_1}{|\psi_1|^2 + |\phi_1|^2} \tag{2.14a}$$

$$\tilde{\mathcal{N}} = \mathcal{N} - 2\frac{\partial}{\partial \eta}\left[(\lambda_1 + \bar{\lambda}_1)\frac{|\psi_1|^2 - |\phi_1|^2}{|\psi_1|^2 + |\phi_1|^2}\right] \tag{2.14b}$$

$$\tilde{\rho} = \rho + 4\frac{\partial}{\partial \eta}\frac{(\lambda_1 + \bar{\lambda}_1)\psi_1 \bar{\phi}_1}{|\psi|^2 + |\phi_1|^2} \tag{2 14c}$$

where $E, \mathcal{N}, \rho$ comprise the seed solution to be found from equation (2.2). The solution for the electric field amplitude (2.14*a*) gives immediately the energy density of the electric field,

$$|\tilde{E}|^2 = |E|^2 + 4\tilde{c}(\lambda_1 + \bar{\lambda}_1)\frac{\partial}{\partial \xi}\ln(|\psi_1|^2 + |\phi_1|^2) + 4\frac{\partial^2}{\partial \xi^2}\ln(|\psi_1|^2 + |\phi_1|^2). \tag{2.15}$$

It is straightforward to work out the Darboux dressing equations for an N-step iteration. We give these equations in the appendix.

Thanks to its algebraic nature the DT method allows wide classes of solutions to be built up, generally in an arbitrary background, which is advantageous in the explicit computations. As far as the MB system is concerned, some examples of successful application of the DT formalism can be found in [4, 5, 8] where soliton solutions in periodic and constant backgrounds are considered. In a forthcoming publication we will consider the question of an optical pulse with a background of a finitely-many gaps potential by Darboux dressing of the corresponding Ψ-function [6]. The question of the interaction of an optical pulse with a spontaneous radiation field is investigated by means of the DT method in [8].

The simplest seed solution of equation (2.2) is that for zero electric field,

$$\begin{aligned} &E=0 \qquad \rho=0 \\ &\mathcal{N}(\xi,\eta)=\mathcal{N}_0 e^{-\tilde{c}\xi}+\frac{4c_0}{\tilde{c}}(e^{-\tilde{c}\xi}-e^{-2\tilde{c}\xi}) \end{aligned} \tag{2.16}$$

When this seed solution is used in the auxiliary linear problem (2.4), we find that

$$\psi_1=c_1 e^{\theta(\xi,\eta)} \qquad \phi_1=c_2 e^{-\theta(\xi,\eta)} \tag{2.17}$$

where $c_{1,2}$ are constants and

$$\theta(\xi,\eta)=\frac{\mathcal{N}_0}{4c_0}[(2c_0\eta+z_1)^{1/2}-z_1{}^{1/2}]+\tilde{c}^{-1}(2c_0\eta+z_1)^{1/2}(1-e^{-\tilde{c}\xi}). \tag{2.18}$$

The result (2.14a) now means that a non-zero electric field amplitude satisfying equations (2.2) is

$$\tilde{E}(\xi,\eta)=2(\lambda_1+\bar{\lambda}_1)\frac{\exp(\theta(\xi,\eta)-\bar{\theta}(\xi,\eta)+i\phi_0)}{\cosh(\theta(\xi,\eta)+\bar{\theta}(\xi,\eta)+\psi_0)}. \tag{2.19}$$

Corresponding expressions for the polarization and inversion are easily found from (2.14b) and (2.14c)

$$\tilde{\rho}=\frac{\lambda_1+\bar{\lambda}_1}{|\lambda_1|^2}\frac{\exp(\theta-\bar{\theta}+i\phi_0)}{\cosh(\theta+\bar{\theta}+\psi_0)}\left\{4c-(\lambda_1-\bar{\lambda}_1)\mathcal{N}-(\lambda_1+\bar{\lambda}_1)\frac{\mathcal{N}}{\cosh(\theta+\bar{\theta}+\psi_0)}\right\} \tag{2.19a}$$

$$\tilde{\mathcal{N}}=\mathcal{N}-2\frac{\lambda_1+\bar{\lambda}_1}{|\lambda_1|^2}\left\{c\tanh(\theta+\bar{\theta}+\psi_0)+\frac{\mathcal{N}(\lambda_1+\bar{\lambda}_1)}{4\cosh^2(\theta+\bar{\theta}+\psi_0)}\right\}. \tag{2.19b}$$

Here $\mathcal{N}$ has the value (2.16).

It is easy to show that for $c_0=\tilde{c}=0$ solution (2.19) is the usual one-soliton solution of SIT in the case of an infinitely narrow line width [7]. Indeed, taking $z_1{}^{1/2}=\alpha+i\beta$, we get

$$\tilde{E}(\xi,\eta)=4\alpha\exp -i\left(\frac{\beta\mathcal{N}_0}{2(\alpha^2+\beta^2)}\eta-\phi_0\right)\left[\cosh 2\alpha\left(\xi-\xi_0+\frac{\mathcal{N}_0}{4(\alpha^2+\beta^2)}\eta\right)\right]^{-1}.$$

In order to analyse the solution (2 19) in more detail we shall make the transformation to the real space variables x, t:

$$\xi = \Omega\left(t - \frac{x}{c}\right) \qquad \eta = \frac{\Omega}{c}x \tag{2 20}$$

where $\Omega^2 = 2\pi\hbar^{-1}n_0\omega_0\mu^2$, and c is the velocity of light. Here n_0 is the density of the two-level atoms, $\hbar\omega_0$ is the energy difference between the levels and μ is the dipole momentum. We then find from (2.19) that, asymptotically when $t \to \infty$, the group velocity v of the electromagnetic pulse behaves as

$$v = c + \frac{cA\Omega^{-3/2}}{4\sqrt{2c_0}(1 + (\tilde{c}N_0/4c_0))} t^{-3/2} + \mathrm{o}(t^{-3/2}). \tag{2.21}$$

The group velocity of the pulse approaches thus the velocity of light In the same limit $t \to \infty$, the width in space Δx of the pulse is asymptotically

$$\Delta x = \frac{c\chi^{(1)}\Omega^{-3/2}}{\sqrt{2c_0}(1 + (\tilde{c}N_0/4c_0))} t^{-1/2} + \mathrm{o}(t^{-1/2}) \tag{2.22}$$

where $\chi^{(1)}$ is the fixed phase $\chi^{(1)} = \theta + \bar{\theta} + \psi_0$. Similarly the point of maximum amplitude of the pulse (2 19) moves in the xt plane along the asymptotic trajectory

$$x - ct = \frac{c}{\tilde{c}\Omega}\left(1 + \frac{\tilde{c}N_0}{4c_0}\right) - \frac{cA\Omega^{-3/2}}{2\sqrt{2c_0}(1 + (\tilde{c}N_0/4c_0))} t^{-1/2} + \mathrm{o}(t^{-1/2}). \tag{2.23}$$

Finally it should be noted that, in the limit $t \to \infty$, the exponential factor $\exp(-\tilde{c}\xi) = \exp(-\tilde{c}\Omega(t - (x/c)))$ tends to one and, therefore, the amplitude of the pulse (2.19) diverges as $t^{1/2}$

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Appendix

In this appendix we derive the Darboux dressing equations for an N-step iteration. To this end we have to generalize ansatz (1.8) such that we assume

$$\Phi[N] = \sum_{j=0}^{N} S_j \Phi \Lambda^{N-j} S_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}. \tag{A1}$$

One can now proceed in the same way as in section 2 for the one-step iteration. Inserting ansatz (A1) into equation (2.4) we find that the following relationships must be satisfied:

$$\begin{aligned} &S_0 U_1 = U_1 S_0 \\ &U_0[N] = S_N U_0 S_N^{-1} + S_{N\xi} S_N^{-1} \qquad 0 \leqslant j \leqslant N-1 \\ &U_0[N] = S_j U_0 S_j^{-1} + S_{j\xi} S_j^{-1} - \tilde{c}(N-j)I + [S_{j+1}, U_1] S_j^{-1} \end{aligned} \tag{A2}$$

and

$$S_{0\eta}=0$$
$$\hat{\rho}[N]=S_0\hat{\rho}S_0^{-1}+4S_{1\eta}S_0^{-1}$$
$$\hat{\rho}[N]=S_N\hat{\rho}S_N^{-1}+4cS_{N-1}S_N^{-1} \tag{A3}$$
$$\hat{\rho}[N]=S_{j-1}\hat{\rho}S_{j-1}^{-1}+4S_{j\eta}S_{j-1}^{-1}+4c(N-j+2)S_{j-2}S_{j-1}^{-1} \qquad 2\leqslant j\leqslant N.$$

These equations will determine the dressed variables in terms of matrices S_j which we still have to define.

First we choose a set $\{\Phi_m,\Lambda_m\}_{m=1}^N$,

$$\Phi_m=\begin{pmatrix}\psi_{2m-1} & \psi_{2m}\\ \phi_{2m-1} & \phi_{2m}\end{pmatrix} \qquad \Lambda_m=\begin{pmatrix}\lambda_{2m-1} & 0\\ 0 & \lambda_{2m}\end{pmatrix} \tag{A4}$$

of solutions of the auxiliary linear problem. As before Λ_m is the matrix spectral parameter related to Φ_m. We then set

$$\tilde{\Phi}[N]|_{\Lambda=\Lambda_m,\Phi=\Phi_m}=0 \qquad m=1,\ldots,N. \tag{A5}$$

In terms of the solutions (A4) equation (A5) can be expressed in the form

$$\sum_{j=1}^{N}\lambda_n^{N-j}(a_j\psi_n+b_j\phi_n)=-\lambda_n^N\psi_n$$
$$\text{for} \qquad n=1,2,\ldots,2N \tag{A6}$$
$$\sum_{j=1}^{N}\lambda_n^{N-j}(c_j\psi_n+d_j\phi_n)=-\lambda_n^N\phi_n.$$

In deriving equation (A6) we have used the result

$$S_0=\mathrm{diag}(a_0,d_0)\mathrm{e}^{\bar{c}N\xi} \tag{A7}$$

which can be easily found from equation (A2). Because S_0 is only a normalizing factor we can choose it to be

$$S_0=\mathrm{I}\cdot\mathrm{e}^{\bar{c}N\xi}. \tag{A8}$$

Equation (A6) can be solved by, for example, the Cramer's rule and we find

$$a_j=-\frac{\det(v_{nl}^{(2j-1)})}{\det(U_{nl})} \qquad b_j=-\frac{\det(v_{nl}^{(2j)})}{\det(U_{nl})}$$
$$c_j=-\frac{\det(w_{nl}^{(2j-1)})}{\det(U_{nl})} \qquad d_j=-\frac{\det(w_{nl}^{(2j)})}{\det(U_{nl})} \qquad j=1,2,\ldots,N \tag{A9}$$

where

$$U_{nl}=\begin{cases}\lambda_n^{N-k}\psi_n & l=2k-1\\ \lambda_n^{N-k}\phi_n & l=2k\end{cases} \qquad \text{for} \qquad k=1,2,\ldots,N$$

$$v_{nl}^{(i)}=\begin{cases}U_{nl} & l\neq i\\ \lambda_n^N\psi_n & l=i\end{cases}$$
$$\text{for} \qquad i=1,2,\ldots,2N \tag{A10}$$
$$w_{nl}^{(i)}=\begin{cases}U_{nl} & l\neq i\\ \lambda_n^N\phi_n & l=i.\end{cases}$$

In order to save the involution property of the system (2.4) together with (2 5), we should choose, as in expression (2.19), the matrix Φ_m of the fixed solutions of the auxiliary problem in the form

$$\Phi_m = \begin{pmatrix} \psi_{2m-1} & -\bar{\phi}_{2m-1} \\ \phi_{2m-1} & \bar{\psi}_{2m-1} \end{pmatrix} \tag{A11}$$

and the associated matrix spectral parameter in the form

$$\Lambda_m = \mathrm{diag}(\lambda_{2m-1} \qquad - \bar{\lambda}_{2m-1}). \tag{A12}$$

References

[1] Burtsev S P, Zakharov V E and Mikhailov A V 1987 *Theor. Math Phys* 70 227; 1987 *Teor Mat. Fiz.* 70 323

[2] Salle M A 1982 *Theor. Math. Phys.* 53 1092, 1982 *Teor. Math. Fiz.* 53 227

[3] Matveev V B 1979 *Lett. Math. Phys* 3 213, 217, 425, 503

[4] Rybin A V 1989 *PhD Thesis* University of Leningrad

[5] Matveev V B, Salle M B and Rybin A V 1988 *Inverse Problems* 4 173–83

[6] Korotkin D A and Matveev V B 1988 *Proc. XVI Workshop on Interdisciplinary Study of Inverse Problems (Some Topics on Inverse Problems)* ed P Sabatier (Singapore. World Scientific)

[7] Ablowitz M J, Kaup D J and Newell A C 1974 *J. Math. Phys.* 15 1852

[8] Rybin A V and Timonen J *Preprint* Department of Physics, University of Jyväskylä (to appear)

[9] Levi D and Ragnisco O 1988 *Inverse Problems* 4 815–28

[10] Neugebauer G and Meinel R 1984 *Phys. Lett* 100A N9

[11] Calogero P and Degasperis A 1978 *Lett. Nuovo Cimento* 23 143, 150

[12] Newell A C 1979 *Proc. R. Soc.* A 365 283–311